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THE MOST RESTRICTIVE BOUNDS ON CHANGE IN THE APPLIED ELASTIC CONSTANTS FOR ANISOTROPIC MATERIALS

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A representation of the elastic constants tensor was given in [1], for the general anisotropic case, in a form which ensured positive definiteness of the specific strain energy and which indicates the strictest bounds for each elastic constant. In the same paper, there is reference to works in which this question had been previously addressed, and which studied the general properties of the tensor of elastic constants. The limits of the variability of the elastic constants was also studied in [2, 3]. Formulas for the characteristic elastic moduli and the states for materials of all crystallographic systems were obtained in [4].

In this work, explicit formulas for the applied elastic constants (Young's modulus, the shear and bulk moduli, Poisson's ratio) are given on the basis of the representation from [1] for the general anisotropic case. The formulas show the limits of variability of these constants. The appropriate formulas for the elastic constants for materials of all crystallographic systems are given. The strictest bounds (without refinement) on these constants which ensure a positive definite specific strain energy are established.

1. In the matrix notation of [1, 4], Hooke's law and the specific strain energy are written as

$$\sigma_i = A_{ij}\varepsilon_j,\ \varepsilon_i = a_{ij}\sigma_j; \tag{1.1}$$

$$2\Phi = \sigma_i\varepsilon_i = A_{ij}\varepsilon_i\varepsilon_j = a_{ij}\sigma_i\sigma_j. \tag{1.2}$$

Here and below, repeated indices denote summation from 1 to 6. The matrices of the elastic constants A_{ij} and a_{ij} are symmetric, and the quadratic form (1.2) is positive definite.

As shown in [1], a_{ij} and A_{ij} can be represented in the form

$$a_{ij} = d_1c_{i1}c_{j1} + d_2c_{i2}c_{j2} + d_3c_{i3}c_{j3} + d_4c_{i4}c_{j4} + d_5c_{i5}c_{j5} + d_6c_{i6}c_{j6}, \quad c_{ip} = 0\ (p > i),\ c_{11} = \ldots = c_{66} = 1; \tag{1.3}$$

$$A_{ij} = d_1^{-1}c_{1i}^{-1}c_{1j}^{-1} + d_2^{-1}c_{2i}^{-1}c_{2j}^{-1} + d_3^{-1}c_{3i}^{-1}c_{3j}^{-1} + d_4^{-1}c_{4i}^{-1}c_{4j}^{-1} + d_5^{-1}c_{5i}^{-1}c_{5j}^{-1} + d_6^{-1}c_{6i}^{-1}c_{6j}^{-1}, \quad c_{ip}^{-1} = 0\quad (p > i),\quad c_{11}^{-1} = \ldots = c_{66}^{-1} = 1. \tag{1.4}$$

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In this case, in (1.3) and (1.4) we have

$$d_1>0,\ d_2>0,\ d_3>0,\ d_4>0,\ d_5>0,\ d_6>0 \quad (1.5)$$

and the elements of the matrix c_{ij}^{-1} are computed using the recursion formula

$$c_{ij}^{-1}=\delta_{ij}-c_{ik}c_{kj}^{-1}\quad (k\leqslant i-1,\ i=1,\ldots,6) \quad (1.6)$$

(δ_{ij} is the unit matrix).

Condition (1.5) is necessary and sufficient for positive definiteness of the matrices a_{ij} and A_{ij} [1]. By assigning six positive numbers d_k and 15 arbitrary parameters c_{ik} ($i > k$) according to (1.3), (1.4), and (1.6), we obtain the limits of change for each constant a_{ij} or A_{ij} for all anisotropic materials which can be described solely in terms of Hooke's law (1.1) (cf. for example, [3, 5]). Formulas (1.4) and (1.6) make it possible simply to find A_{ij}, that is, the inverse matrix to a_{ij}. If a_{ij} is given, then from (1.3) it is straightforward to find d_k, c_{ik} [1], and to verify (1.5). Formulas (1.3) and (1.4) are completely equivalent in the sense that if A_{ij} can be represented as (1.3), then a_{ij} will have the form (1.4).

Using (1.3) and (1.4), we write the specific strain energy (1.2) in the following way:

$$\begin{aligned}2\Phi &= d_1(\sigma_1+c_{21}\sigma_2+c_{31}\sigma_3+c_{41}\sigma_4+c_{51}\sigma_5+c_{61}\sigma_6)^2+d_2(\sigma_2+c_{32}\sigma_3+\\&+c_{42}\sigma_4+c_{52}\sigma_5+c_{62}\sigma_6)^2+d_3(\sigma_3+c_{43}\sigma_4+c_{53}\sigma_5+c_{63}\sigma_6)^2+\\&+d_4(\sigma_4+c_{54}\sigma_5+c_{64}\sigma_6)^2+d_5(\sigma_5+c_{65}\sigma_6)^2+d_6\sigma_6^2=\\&=d_1^{-1}\varepsilon_1^2+d_2^{-1}(c_{21}^{-1}\varepsilon_1+\varepsilon_2)^2+d_3^{-1}(c_{31}^{-1}\varepsilon_1+c_{32}^{-1}\varepsilon_2+\varepsilon_3)^2+\\&+d_4^{-1}(c_{41}^{-1}\varepsilon_1+c_{42}^{-1}\varepsilon_2+c_{43}^{-1}\varepsilon_3+\varepsilon_4)^2+d_5^{-1}(c_{51}^{-1}\varepsilon_1+c_{52}^{-1}\varepsilon_2+c_{53}^{-1}\varepsilon_3+\\&+c_{54}^{-1}\varepsilon_4+\varepsilon_5)^2+d_6^{-1}(c_{61}^{-1}\varepsilon_1+c_{62}^{-1}\varepsilon_2+c_{63}^{-1}\varepsilon_3+c_{64}^{-1}\varepsilon_4+c_{65}^{-1}\varepsilon_5+\varepsilon_6)^2.\end{aligned} \quad (1.7)$$

In place of a_{ij} one can use the technical notaiton [6]:

$$a_{ij}=\nu_{ij}/E_j=\nu_{ji}/E_i,\quad \nu_{11}=\ldots=\nu_{66}=1 \quad (1.8)$$

(there is no summation over i or j). Here E_1, E_2, E_3 are the Young's moduli in the directions of the axes; $E_4 = 2\mu_{23}$, $E_5 = 2\mu_{13}$, $E_6 = 2\mu_{12}$ are the shear moduli; ν_{ij} (i, j = 1, 2, 3) are Poisson's ratios; ν_{ij} (i, j = 4, 5, 6) are Chentsov coefficients; ν_{ij} (i = 1, 2, 3, j = 4, 5, 6 or j = 1, 2, 3, i = 4, 5, 6) are the interactive influence coefficients. In (1.8), in contrast to the traditional notation, there is no minus sign in front of Poisson's ratio. In our opinion, this is more natural.

Let n_i, m_i (i = 1, 2, 3) be two orthogonal directions: $n_in_i = 1$, $m_im_i = 1$, and $n_im_i = 0$. We use the notation

$$\begin{aligned}(n)_i&=(n_1^2,n_2^2,n_3^2,\ \sqrt{2}\,n_2n_3,\ \sqrt{2}\,n_1n_3,\ \sqrt{2}\,n_1n_2),\\(m)_i&=(m_1^2,m_2^2,m_3^2,\ \sqrt{2}\,m_2m_3,\ \sqrt{2}\,m_1m_3,\ \sqrt{2}\,m_1m_2),\\(nm)_i&=(n_1m_1,\ n_2m_2,\ n_3m_3,\ \sqrt{2}(n_2m_3+n_3m_2)/2,\\&\quad\sqrt{2}(n_1m_3+n_3m_1)/2,\ \sqrt{2}(n_1m_2+n_2m_1)/2).\end{aligned} \quad (1.9)$$

Now, using formuals (1.3) and (1.9), we will obtain the applied elastic constants. We write the bulk modulus as

$$1/K=a_{11}+a_{22}+a_{33}+2(a_{21}+a_{31}+a_{32})=d_1(1+c_{21}+c_{31})^2+d_2(1+c_{32})^2+d_3. \quad (1.10)$$

From (1.10) it is clear that the bulk modulus is always positive (compare with [6]). Young's modulus in the direction n_i is represented as:

$$\begin{aligned}1/E_n&=(n)_i\,a_{ij}\,(n)_j=d_1(n_1^2+n_2^2c_{21}+n_3^2c_{31}+\sqrt{2}\,n_2n_3c_{41}+\\&+\sqrt{2}\,n_1n_3c_{51}+\sqrt{2}\,n_1n_2c_{61})^2+d_2(n_2^2+n_3^2c_{32}+\sqrt{2}\,n_2n_3c_{42}+\sqrt{2}\,n_1n_3c_{52}+\\&+\sqrt{2}\,n_1n_2c_{62})^2+d_3(n_3^2+\sqrt{2}n_2n_3c_{43}+\sqrt{2}\,n_1n_3c_{53}+\sqrt{2}\,n_1n_2c_{63})^2+\\&+2d_4(n_2n_3+n_1n_3c_{54}+n_1n_2c_{64})^2+2d_5(n_1n_3+n_1n_2c_{65})^2+2d_6(n_1n_2)^2.\end{aligned} \quad (1.11)$$

Poisson's ratio in the direction m_i with tension in the direction n_i has the form

$$\nu_{mn}/E_n=(m)_i\,a_{ij}\,(n)_j=d_1(m_1^2+m_2^2c_{21}+m_3^2c_{31}+\sqrt{2}\,m_2m_3c_{41}+$$

$$+ \sqrt{2}\, m_1 m_3 c_{51} + \sqrt{2}\, m_1 m_2 c_{61}\big)\big(n_1^2 + n_2^2 c_{21} + n_3^2 c_{31} + \sqrt{2}\, n_2 n_3 c_{41} +$$
$$+ \sqrt{2}\, n_1 n_3 c_{51} + \sqrt{2}\, n_1 n_2 c_{61}\big) + d_2\big(m_2^2 + m_3^2 c_{32} + \sqrt{2}\, m_2 m_3 c_{42} +$$
$$+ \sqrt{2}\, m_1 m_3 c_{52} + \sqrt{2}\, m_1 m_2 c_{62}\big)\big(n_2^2 + n_3^2 c_{32} + \sqrt{2}\, n_2 n_3 c_{42} + \sqrt{2}\, n_1 n_3 c_{52} +$$
$$+ \sqrt{2}\, n_1 n_2 c_{62}\big) + d_3\big(m_3^2 + \sqrt{2}\, m_2 m_3 c_{43} + \sqrt{2}\, m_1 m_3 c_{53} + \sqrt{2}\, m_1 m_2 c_{63}\big)\big(n_3^2 +$$
$$+ \sqrt{2}\, n_2 n_3 c_{43} + \sqrt{2}\, n_1 n_3 c_{53} + \sqrt{2}\, n_1 n_2 c_{63}\big) + 2d_4\,(m_2 m_3 + m_1 m_3 c_{54} +$$
$$+ m_1 m_2 c_{64})\,(n_2 n_3 + n_1 n_3 c_{54} + n_1 n_2 c_{64}) + 2d_5\,(m_1 m_3 + m_1 m_2 c_{65})(n_1 n_3 + n_1 n_2 c_{65}) + 2d_6 m_1 m_2 n_1 n_2. \qquad (1.12)$$

From (1.11) and (1.12) we obtain

$$E_1^{-1} = d_1 = a_{11},\ E_2^{-1} = d_1 c_{21}^2 + d_2 = a_{22},\ E_3^{-1} = d_1 c_{31}^2 + d_2 c_{32}^2 + d_3 = a_{33},\ \nu_{21} = c_{21},\ \nu_{31} = c_{31},$$
$$\nu_{12} = \frac{d_1 c_{21}}{d_1 c_{21}^2 + d_2},\quad \nu_{32} = \frac{d_1 c_{31} c_{21} + d_2 c_{32}}{d_1 c_{21}^2 + d_2}, \qquad (1.13)$$
$$\nu_{13} = \frac{d_1 c_{31}}{d_1 c_{31}^2 + d_2 c_{32}^2 + d_3},\quad \nu_{23} = \frac{d_1 c_{31} c_{21} + d_2 c_{32}}{d_1 c_{31}^2 + d_2 c_{32}^2 + d_3}.$$

The shear modulus between surfaces with normals n_i and m_i is written as

$$1/(4\mu_{nm}) = (nm)_i a_{ij} (nm)_j = d_1[n_1 m_1 + n_2 m_2 c_{21} + n_3 m_3 c_{31} + (\sqrt{2}/2)(n_2 m_3 +$$
$$+ n_3 m_2)c_{41} + (\sqrt{2}/2)(n_1 m_3 + n_3 m_1)c_{51} + (\sqrt{2}/2)(n_1 m_2 + n_2 m_1)c_{61}]^2 + d_2[n_2 m_2 + n_3 m_3 c_{32} + (\sqrt{2}/2)(n_2 m_3 +$$
$$+ n_3 m_2)c_{42} + (\sqrt{2}/2)(n_1 m_3 + n_3 m_1)c_{52} + (\sqrt{2}/2)(n_1 m_2 + n_2 m_1)c_{62}]^2 + d_3[n_3 m_3 + (\sqrt{2}/2)(n_2 m_3 + n_3 m_2)c_{43} +$$
$$+ (\sqrt{2}/2)(n_1 m_3 + n_3 m_1)c_{53} + (\sqrt{2}/2)(n_1 m_2 + n_2 m_1)c_{63}]^2 + (1/2)d_4[(n_2 m_3 + n_3 m_2) + (n_1 m_3 + n_3 m_1)c_{54} + (n_1 m_2 +$$
$$+ n_2 m_1)c_{64}]^2 + (1/2)d_5[(n_1 m_3 + n_3 m_1) + (n_1 m_2 + n_2 m_1)c_{65}]^2 + (1/2)d_6(n_1 m_2 + n_2 m_1)^2. \qquad (1.14)$$

From (1.14) we have

$$1/(2\mu_{23}) = d_1 c_{41}^2 + d_2 c_{42}^2 + d_3 c_{43}^2 + d_4 = a_{44},$$
$$1/(2\mu_{13}) = d_1 c_{51}^2 + d_2 c_{52}^2 + d_3 c_{53}^2 + d_4 c_{54}^2 + d_5 = a_{55}, \qquad (1.15)$$
$$1/(2\mu_{12}) = d_1 c_{61}^2 + d_2 c_{62}^2 + d_3 c_{63}^2 + d_4 c_{64}^2 + d_5 c_{65}^2 + d_6 = a_{66}.$$

By assigning arbitrary values to the parameters $d_k > 0$, c_{ik} $(i > k)$, n_i, m_i in (1.10)-(1.15), we obtain the admissible limits on the change of the corresponding elastic constants for an arbitrary anisotropic material.

2. We now consider materials which have symmetry properties with respect to orthogonal transformations of the coordinate system. These materials (crystals) are divided into seven systems and isotropic media, according to their symmetry properties [7]. The matrices a_{ij}, A_{ij} will be written out for materials is these crystallographic systems, following [4, 7].

We write these matrices in the form (1.3), (1.4), and the specific strain energy in the form (1.7). It will be obvious from this what the matrices c_{ik}, c_{ik}^{-1} $(i > k)$ are equal to for each system.

Cubic System

$$a_{ij} = \begin{bmatrix} d_1 & a_{21} & a_{21} & 0 & 0 & 0 \\ d_1 c_{21} & a_{11} & a_{21} & 0 & 0 & 0 \\ a_{21} & a_{21} & a_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{44} \end{bmatrix},\quad A_{ij} = \begin{bmatrix} \dfrac{1 + c_{21}}{d_1(1 - c_{21})(1 + 2c_{21})} & A_{21} & A_{21} & 0 & 0 & 0 \\ \dfrac{-c_{21}}{d_1(1 - c_{21})(1 + 2c_{21})} & A_{11} & A_{21} & 0 & 0 & 0 \\ A_{21} & A_{21} & A_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{44} \end{bmatrix},$$

$$2\Phi = d_1[\sigma_1 + c_{21}(\sigma_2 + \sigma_3)]^2 + d_2\left(\sigma_2 + \frac{c_{21}}{1 + c_{21}}\sigma_3\right)^2 + d_3\sigma_3^2 + d_4(\sigma_4^2 + \sigma_5^2 + \sigma_6^2) =$$

$$= d_1^{-1}\varepsilon_1^2 + d_2^{-1}(-c_{21}\varepsilon_1 + \varepsilon_2)^2 + d_3^{-1}\left[\frac{-c_{21}}{1+c_{21}}(\varepsilon_1+\varepsilon_2)+\varepsilon_3\right]^2 + d_4^{-1}(\varepsilon_4^2+\varepsilon_5^2+\varepsilon_6^2), \quad d_2 = d_1(1-c_{21}^2),$$

$$d_3 = d_1(1-c_{21})\frac{1+2c_{21}}{1+c_{21}}, \; d_5 = d_6 = d_4. \tag{2.1}$$

It is clear from these formulas that cubic system materials are determined by three independent parameters: d_1, d_4, and c_{21}, and from (1.5) it follows that

$$d_1 > 0, \; d_4 > 0, \; -1/2 < c_{21} < 1. \tag{2.2}$$

The applied constants for cubic system materials are

$$\begin{aligned} 1/K &= 3d_1(1+2c_{21}),\\ 1/E_n &= d_1 + [d_4 - d_1(1-c_{21})][1-(n_1^4+n_2^4+n_3^4)],\\ \nu_{mn}/E_n &= d_1c_{21} - [d_4 - d_1(1-c_{21})][(n_1m_1)^2 + (n_2m_2)^2 + (n_3m_3)^2],\\ 1/(4\mu_{nm}) &= (1/2)d_4 - [d_4 - d_1(1-c_{21})][(n_1m_1)^2 + (n_2m_2)^2 + (n_3m_3)^2]. \end{aligned} \tag{2.3}$$

To change to an isotropic material we set $d_4 = d_1(1 - c_{21})$ in (2.1) and (2.3), and (2.2) is retained.

Trigonal System

$$a_{ij} = \begin{bmatrix} d_1 & a_{21} & a_{31} & a_{41} & 0 & 0\\ d_1c_{21} & a_{11} & a_{31} & -a_{41} & 0 & 0\\ d_1c_{31} & a_{31} & d_1\frac{2c_{31}^2}{1+c_{21}}+d_3 & 0 & 0 & 0\\ d_1c_{41} & -a_{41} & 0 & d_1\frac{2c_{41}^2}{1-c_{21}}+d_4 & 0 & 0\\ 0 & 0 & 0 & 0 & a_{44} & \sqrt{2}a_{41}\\ 0 & 0 & 0 & 0 & \sqrt{2}a_{41} & d_1(1-c_{21}) \end{bmatrix},$$

$$A_{ij} = \begin{bmatrix} \frac{1}{d_1(1-c_{21}^2)}+\frac{c_{31}^2}{d_3(1+c_{21})^2}+\frac{c_{41}^2}{d_4(1-c_{21})^2} & A_{21} & A_{31} & A_{41} & 0 & 0\\ \frac{-c_{21}}{d_1(1-c_{21}^2)}+\frac{c_{31}^2}{d_3(1+c_{21})^2}-\frac{c_{41}^2}{d_4(1-c_{21})^2} & A_{11} & A_{31} & -A_{41} & 0 & 0\\ \frac{-c_{31}}{d_3(1+c_{21})} & A_{31} & d_3^{-1} & 0 & 0 & 0\\ \frac{-c_{41}}{d_4(1-c_{21})} & -A_{41} & 0 & d_4^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & A_{44} & \sqrt{2}A_{41}\\ 0 & 0 & 0 & 0 & \sqrt{2}A_{41} & \frac{1}{d_1(1-c_{21})}+\frac{2c_{41}^2}{d_4(1-c_{21})^2} \end{bmatrix}, \tag{2.4}$$

$$\begin{aligned} 2\Phi &= d_1(\sigma_1 + c_{21}\sigma_2 + c_{31}\sigma_3 + c_{41}\sigma_4)^2 + d_2\left(\sigma_2 + \frac{c_{31}}{1+c_{21}}\sigma_3 - \frac{c_{41}}{1-c_{21}}\sigma_4\right)^2 +\\ &+ d_3\sigma_3^2 + d_4\sigma_4^2 + d_5\left[\sigma_5 + \frac{\sqrt{2}\,d_1c_{41}(1-c_{21})}{d_1 2c_{41}^2 + d_4(1-c_{21})}\sigma_6\right]^2 + d_6\sigma_6^2 = d_1^{-1}\varepsilon_1^2 +\\ &+ d_2^{-1}(-c_{21}\varepsilon_1+\varepsilon_2)^2 + d_3^{-1}\left[\frac{-c_{31}}{1+c_{21}}(\varepsilon_1+\varepsilon_2)+\varepsilon_3\right]^2 + d_4^{-1}\left[\frac{-c_{41}}{1-c_{21}}(\varepsilon_1-\varepsilon_2)+\varepsilon_4\right]^2 +\\ &+ d_5^{-1}\varepsilon_5^2 + d_6^{-1}\left[\frac{-\sqrt{2}\,d_1c_{41}(1-c_{21})}{d_1 2c_{41}^2 + d_4(1-c_{21})}\varepsilon_5 + \varepsilon_6\right]^2, \end{aligned}$$

$$d_2 = d_1(1-c_{21}^2), \; d_5 = d_1\frac{2c_{41}^2}{1-c_{21}} + d_4,$$

$$d_6 = \frac{d_1d_4(1-c_{21})^2}{d_1 2c_{41}^2 + d_4(1-c_{21})}.$$

From (2.4) it is clear that the trigonal system material is given by six independent parameters d_1, d_3, d_4, c_{21}, c_{31}, and c_{41}, with

$$d_1 > 0,\ d_3 > 0,\ d_4 > 0,\ -1 < c_{21} < 1. \tag{2.5}$$

Condition (2.5) ensures that the specific strain energy is positive definite for trigonal system materials. It follows from (2.5) that the lower bound on Poisson's coefficient ν_{21} is −1 for trigonal systems, and not −1/2, as is the case for isotropic and cubic systems [see (2.2)]. There are no bounds on c_{31} and c_{41}: they can be arbitrary.

If in place of d_3 and d_4, the moduli E_3 and $E_4 = 2\mu_{23}$ are taken as the independent parameters, then from (2.5) and the technical notation (1.8), we have

$$\frac{1}{E_3} > \frac{2\nu_{31}^2}{E_1(1+\nu_{21})},\quad \frac{1}{E_4} > \frac{2\nu_{41}^2}{E_1(1-\nu_{21})}$$

or after transformation

$$-\sqrt{\frac{E_1}{2E_3}(1+\nu_{21})} < \nu_{31} < \sqrt{\frac{E_1}{2E_3}(1+\nu_{21})} = f_1; \tag{2.6}$$

$$-\sqrt{\frac{E_1}{2E_4}(1-\nu_{21})} < \nu_{41} < \sqrt{\frac{E_1}{2E_4}(1-\nu_{21})} = f_2. \tag{2.7}$$

Inequalities (2.6) and (2.7) determine the strictest bounds for change in ν_{31}, ν_{41} as functions of the ratios E_1/E_3, E_1/E_4 and of ν_{21}. The region of permissible values for ν_{31}, ν_{41}, and ν_{21} are shown in Figs. 1 and 2 for $E_1 = 4E_3$, and $E_1 = 2E_4$ (shaded areas).

The applied constants for trigonal system materials are

$$\begin{aligned}
\frac{1}{K} &= 2d_1\frac{(1+c_{21}+c_{31})^2}{1+c_{21}} + d_3,\\
1/E_n &= d_1(1-n_3^2)^2 + \left(d_1 2c_{31}^2/(1+c_{21}) + d_3\right)n_3^4 +\\
&\quad + 2(d_1c_{31}+d_4)(1-n_3^2)n_3^2 + 2\sqrt{2}\,d_1c_{41}(3n_1^2-n_2^2)n_2n_3,\\
\nu_{mn}/E_n &= d_1c_{21} + d_1(c_{31}-c_{21})(n_3^2+m_3^2) + \left[d_1(1-2c_{31}) + d_1 2c_{31}^2/(1+c_{21}) +\right.\\
&\quad \left. + d_3 - 2d_4\right] n_3^2m_3^2 + \sqrt{2}\,d_1c_{41}[(n_1m_1-n_2m_2)(n_2m_3+n_3m_2) +\\
&\quad + (n_1m_3+n_3m_1)(n_1m_2+n_2m_1)],\\
1/(4\mu_{nm}) &= (1/2)d_1(1-c_{21}) + (1/2)[d_4 - d_1(1-c_{21})](n_3^2+m_3^2) +\\
&\quad + \left[d_1(1-2c_{31}) + d_1 2c_{31}^2/(1+c_{21}) + d_3 - 2d_4\right]n_3^2m_3^2 + \sqrt{2}\,d_1c_{41}\times\\
&\quad \times[(n_1m_1-n_2m_2)(n_2m_3+n_3m_2) + (n_1m_3+n_3m_1)(n_1m_2+n_2m_1)].
\end{aligned} \tag{2.8}$$

To change to a material of the hexagonal system (transversally isotropic), we must set $c_{41} = 0$ in (2.4) and (2.8) while retaining (2.5).

To change to a tetragonal system material, it is necessary to set $c_{41} = 0$ in (2.4), and in place of $a_{66} = d_1(1 - c_{21})$, to write $a_{66} = d_6 > 0$ and consider d_6 as an independent parameter. Evidently tetragonal system materials are determined by six independent parameters: d_1, d_3, d_4, d_6, c_{21}, and c_{31}, and (2.5) is retained. The bulk modulus has the form (2.8). The remaining applied constants for tetragonal systems are:

$$\begin{aligned}
1/E_n &= d_1(1-n_3^2)^2 + \left(d_1 2c_{31}^2/(1+c_{21}) + d_3\right)n_3^4 + 2(d_1c_{31}+d_4)(1-n_3^2)n_3^2 +\\
&\quad + [d_6 - d_1(1-c_{21})]\,2(n_1n_2)^2,\\
\nu_{mn}/E_n &= d_1c_{21} + d_1(c_{31}-c_{21})(n_3^2+m_3^2) + \left[d_1(1-2c_{31}) + d_1 2c_{31}^2/(1+c_{21}) +\right.\\
&\quad \left. + d_3 - 2d_4\right]n_3^2m_3^2 + [d_6 - d_1(1-c_{21})]\,2m_1m_2n_1n_2,\\
1/(4\mu_{nm}) &= (1/2)d_1(1-c_{21}) + (1/2)[d_4 - d_1(1-c_{21})](n_3^2+m_3^2) +\\
&\quad + \left[d_1(1-2c_{31}) + d_1 2c_{31}^2/(1+c_{21}) + d_3 - 2d_4\right]n_3^2m_3^2 +\\
&\quad + [d_6 - d_1(1-c_{21})](1/2)(n_1m_2+n_2m_1)^2.
\end{aligned}$$

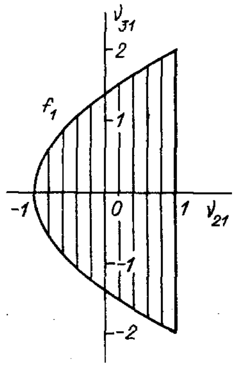

Fig. 1

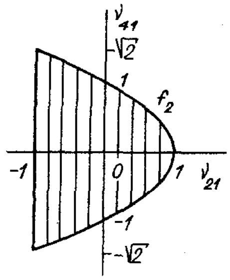

Fig. 2

Orthorhombic System (Orthotropic)

$$a_{ij}=\begin{bmatrix} d_1 & a_{21} & a_{31} & 0 & 0 & 0\\ d_1c_{21} & d_1c_{21}^2+d_2 & a_{32} & 0 & 0 & 0\\ d_1c_{31} & d_1c_{31}c_{21}+d_2c_{32} & d_1c_{31}^2+d_2c_{32}^2+d_3 & 0 & 0 & 0\\ 0 & 0 & 0 & d_4 & 0 & 0\\ 0 & 0 & 0 & 0 & d_5 & 0\\ 0 & 0 & 0 & 0 & 0 & d_6\end{bmatrix},$$

$$A_{ij}=\begin{bmatrix} d_1^{-1}+d_2^{-1}c_{21}^2+d_3^{-1}(-c_{31}+c_{32}c_{21})^2 & A_{21} & A_{31} & 0 & 0 & 0\\ -d_2^{-1}c_{21}-d_3^{-1}c_{32}(-c_{31}+c_{32}c_{21}) & d_2^{-1}+d_3^{-1}c_{32}^2 & A_{32} & 0 & 0 & 0\\ d_3^{-1}(-c_{31}+c_{32}c_{21}) & -d_3^{-1}c_{32} & d_3^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & d_4^{-1} & 0 & 0\\ 0 & 0 & 0 & 0 & d_5^{-1} & 0\\ 0 & 0 & 0 & 0 & 0 & d_6^{-1}\end{bmatrix},$$

$$2\Phi=d_1(\sigma_1+c_{21}\sigma_2+c_{31}\sigma_3)^2+d_2(\sigma_2+c_{32}\sigma_3)^2+d_3\sigma_3^2+d_4\sigma_4^2+d_5\sigma_5^2+d_6\sigma_6^2=$$
$$=d_1^{-1}\varepsilon_1^2+d_2^{-1}(-c_{21}\varepsilon_1+\varepsilon_2)^2+d_3^{-1}[(-c_{31}+c_{32}c_{21})\varepsilon_1-c_{32}\varepsilon_2+\varepsilon_3]^2+d_4^{-1}\varepsilon_4^2+d_5^{-1}\varepsilon_5^2+d_6^{-1}\varepsilon_6^2.$$

It is clear from these formulas that orthotropic materials are determined by nine independent parameters: d_1, $d_2,\ldots,d_6$, c_{21}, c_{31}, and c_{32} which satisfy (1.5). If the moduli E_2 and E_3 are used in place of d_2 and d_3 as the independent parameters, then from (1.5) and the technical notation (1.8), we have

$$\frac{1}{E_2}>\frac{\nu_{21}^2}{E_1},\quad \frac{1}{E_3}>\frac{\nu_{31}^2}{E_1}+\frac{(\nu_{32}/E_2-\nu_{31}\nu_{21}/E_1)^2}{1/E_2-\nu_{21}^2/E_1}. \tag{2.9}$$

The bulk modulus takes the form (1.10). The remaining applied constants for orthotropic materials are:

$$1/E_n=d_1(n_1^2+n_2^2c_{21}+n_3^2c_{31})^2+d_2(n_2^2+n_3^2c_{32})^2+d_3n_3^4+2d_4(n_2n_3)^2+2d_5(n_1n_3)^2+2d_6(n_1n_2)^2,$$
$$\nu_{mn}/E_n=d_1(m_1^2+m_2^2c_{21}+m_3^2c_{31})(n_1^2+n_2^2c_{21}+n_3^2c_{31})+d_2(m_2^2+m_3^2c_{32})(n_2^2+$$
$$+n_3^2c_{32})+d_3m_3^2n_3^2+2d_4m_2m_3n_2n_3+2d_5m_1m_3n_1n_3+2d_6m_1m_2n_1n_2,$$
$$1/(4\mu_{nm})=d_1(n_1m_1+n_2m_2c_{21}+n_3m_3c_{31})^2+d_2(n_2m_2+n_3m_3c_{32})^2+d_3(n_3m_3)^2+d_4(1/2)(n_2m_3+$$
$$+n_3m_2)^2+d_5(1/2)(n_1m_3+n_3m_1)^2+d_6(1/2)(n_1m_2+n_2m_1)^2.$$

If the x_1 axis is taken as the axis of second order symmetry, then we obtain the formulas for a monoclinic system material from the general formulas (1.3), (1.4), (1.7), (1.10)-(1.15) by setting $c_{5i}=0$ ($i=1,\ldots,4$), $c_{6i}=0$ ($i=1,\ldots,5$) in these formulas. In this case, the matrix c_{ki}^{-1} has the form

$$c_{ki}^{-1}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -c_{21} & 1 & 0 & 0 & 0 & 0\\ -c_{31}+c_{32}c_{21} & -c_{32} & 1 & 0 & 0 & 0\\ -c_{41}+c_{42}c_{21}+c_{43}(c_{31}-c_{32}c_{21}) & -c_{42}+c_{43}c_{32} & -c_{43} & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

Evidently a monoclinic system material is determined by 12 independent parameters: $d_1, d_2, \ldots, d_6, c_{21}, c_{31}, c_{41}, c_{32}, c_{42}$, and c_{43} with condition (1.5) being satisfied. If the moduli E_2, E_3, and $E_4 = 2\mu_{23}$ are taken as the independent parameters instead of d_2, d_3, and d_4, then by using (1.5) and the technical notation (1.8), we obtain condition (2.9) and

$$\frac{1}{E_4} > \frac{\nu_{41}^2}{E_1} + \frac{(\nu_{42}/E_2 - \nu_{41}\nu_{21}/E_1)^2}{1/E_2 - \nu_{21}^2/E_1} + \frac{\left[\frac{\nu_{43}}{E_3} - \frac{\nu_{41}\nu_{31}}{E_1} - \frac{(\nu_{42}/E_2 - \nu_{41}\nu_{21}/E_1)(\nu_{32}/E_2 - \nu_{31}\nu_{21}/E_1)}{1/E_2 - \nu_{21}^2/E_1}\right]^2}{\frac{1}{E_3} - \frac{\nu_{31}^2}{E_1} - \frac{(\nu_{32}/E_2 - \nu_{31}\nu_{21}/E_1)^2}{1/E_2 - \nu_{21}^2/E_1}}. \quad (2.10)$$

For triclinic system materials, the matrices a_{ij} and A_{ij} have the general form and therefore the formulas (1.3)-(1.7), (1.10)-(1.15) for the general anisotropic case must be used. The conditions $d_5 > 0$, $d_6 > 0$ can be rewritten similarly to (2.9) and (2.10) by using the technical notation (1.8), but due to the unwieldiness of these formulas, they will not be written out here.

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ENERGY VERSION OF CREEP AND STRESS-RUPTURE STRENGTH THEORY FOR ANISOTROPIC AND ISOTROPIC MATERIALS WHICH DIFFER IN RESISTANCE TO TENSION AND COMPRESSION

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UDC 539.3

A new separate branch in solid mechanics has recently been formed, i.e., creep theory for materials which resist tension and compression differently [1-15]. Intense development of it is connected with considerable engineering applications since light alloys, gray cast irons, polymers, ceramics, composites, and other materials whose creep depends on the type of loading are used extensively in various fields of technology. On the other hand, in published works [16-26] considerable attention is devoted to the mechanics of damaged materials. The majority of the approaches in this field are development and generalization of the Rabotnov concept [27] about a material damage parameter. It is evident that deformation and damage accumulation occur under creep conditions in parallel with each other and they have a reciprocal effect. In order to describe these phenomena it is very convenient to use equations of state in an energy form which make it possible to compare creep analysis with finding the time for failure of a structure. Here in the equations it is necessary to reflect the effect of the form of loading on creep and stress-rupture strength.

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